J Stat Phys (2009) 135: 737–750
DOI 10.1007/s10955-009-9760-y

A New Boundary Problem for the Two Dimensional Navier-Stokes System

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Received: 31 March 2009 / Accepted: 14 May 2009 / Published online: 23 May 2009

Abstract We formulate a new boundary value problem for the 2D Navier-Stokes system on the unit square. Under some suitable assumptions on the initial velocity, we obtain quantitative decay estimates of the Fourier modes of both the vorticity and the velocity. It is found that in one direction the Fourier modes decay exponentially and along the other direction their decay is only power like.

Keywords Navier-Stokes equations

1 Introduction and the Formulation of the Main Results

Consider the two dimensional Navier-Stokes System (NSS) for incompressible fluids. In the absence of external forcing, the vorticity $\omega(t,x,y) = \frac{\partial u_1(t,x,y)}{\partial y} - \frac{\partial u_2(t,x,y)}{\partial x}$ satisfies the equation

$$\frac{\partial \omega}{\partial t} + u_1 \cdot \frac{\partial \omega}{\partial x} + u_2 \cdot \frac{\partial \omega}{\partial y} = \Delta \omega. \tag{1.1}$$

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Here the viscosity is taken to be one and u_1, u_2 are the components of the velocity vector $u=(u_1,u_2)$ which satisfies the incompressibility condition

$$\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}=0. \tag{1.2}$$

In the periodic boundary condition case, Foias and Temam [7] showed that if the initial velocity $u_0 \in H^1$, then the Fourier components of ω decay exponentially for any $t>0$. In the papers by Ferrari and Titi [6] this result was extended to general nonlinear parabolic equations with real analytic nonlinearities (see also Cao, Rammaha and Titi [1] for the case on the sphere). In the paper [11] another proof of the Foias and Temam theorem was given using a geometric trapping argument. In our paper [4] other cases for which the result of [7] remains valid were described. In the whole space case, the first existence and uniqueness theorems for weak solutions of (1.1) were proven by Leray [10] for the NSS and later extended by Hopf [8]. Ladyzenskaya [9] proved the existence and uniqueness results for strong solutions in general two-dimensional bounded domains. We refer the interested readers to [3, 5, 11–14] and references therein for more related results.

It is a natural question whether exponential decay of Fourier modes is exceptional or typical. We believe that it is exceptional and in the general case it is a power-like decay with a power depending on the geometry of the domain. In this paper we describe a new boundary value problem for NSS for which we give upper estimates of Fourier modes. As it will be shown, these estimates decay exponentially in one direction and only power-like in the other direction. Presumably it is hopeless to get general lower estimates but we believe that our upper estimates are close to being optimal. In the accompanying paper by N. Chernov (see [2]) numerical results are given which confirms this type of behavior.

New Boundary Problem Consider NSS inside the two dimensional square Q whose sides are equal to π. The first set of boundary conditions is posed for the velocity component u_1. Namely, assume that

$$u_1(t,x,y)=0, \quad \text{for all } (x,y)\in\partial Q \text{ and all } t\ge 0. \tag{1.3}$$

The second set of boundary conditions is posed in terms of the vorticity. More precisely, we assume on the vertical parts of ∂Q,

$$\omega|_{x=0}=\omega|_{x=\pi}=0, \quad \forall\, 0\le y\le\pi,\ t\ge 0, \tag{1.4}$$

and on the horizontal parts of ∂Q,

$$\partial_y\omega|_{y=0}=\partial_y\omega|_{y=\pi}=0, \quad \forall\, 0\le x\le\pi,\ t\ge 0. \tag{1.5}$$

The system (1.1), (1.2) together with the boundary conditions (1.3)–(1.5) then forms a complete set of equations for which we shall prove the existence and uniqueness of classical solutions.

We consider solutions of NSS with finite energy. Therefore by (1.3) u_1 can be expanded into the following Fourier series:

$$u_1(t,x,y)=\sum_{m,n\ge 1}\frac{h(t,m,n)}{m}\sin mx\sin ny. \tag{1.6}$$

We shall assume that this series and all series below like (1.6) converge fast enough so that formal operations like differentiations are possible. The strong decay of the Fourier coefficients will be proved later. Next we write

$$\frac{\partial u_1(t,x,y)}{\partial x} = \sum_{m,n\geq 1} h(t,m,n)\cos mx \sin ny.$$

From the incompressibility condition

$$\frac{\partial u_2}{\partial y} = -\frac{\partial u_1}{\partial x} = -\sum_{m,n\geq 1} h(t,m,n)\cos mx \sin ny.$$

Therefore we write $u_2(t,x,y)$ in the form

$$u_2(t,x,y) = \sum_{m,n\geq 1} \frac{h(t,m,n)}{n}\cos mx \cos ny + v(t,x), \tag{1.7}$$

where v is a function of (t,x) which we write as a Fourier series

$$v(t,x) = \sum_{m\geq 1} \frac{f(t,m)}{m}\cos mx. \tag{1.8}$$

Note that the term $m=0$ is not included in the sum. The omission of this term is equivalent to the condition that $\int_0^\pi v(t,x)dx = 0$. Thus the basic parameters in our problem are the coefficients $h(t,m,n)$, $m\geq 1$, $n\geq 1$ and $f(t,m)$, $m\geq 1$.

System of ODE for $h(t,m,n)$, $f(t,m)$ For the vorticity $\omega(t,x,y)$ we can write

$$\begin{aligned}\omega &= \frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x}\\ &= \sum_{m\geq 1, n\geq 1} h(t,m,n)\cdot\frac{m^2+n^2}{mn}\cdot \sin mx \cos ny + \sum_{m\geq 1} f(t,m)\sin mx.\end{aligned} \tag{1.9}$$

By using (1.6) and (1.9), we have

$$\begin{aligned}u_1\cdot\frac{\partial\omega}{\partial x} &= \sum_{m'\geq 1, n'\geq 1} h(t,m',n')\cdot\frac{(m')^2+(n')^2}{n'}\cdot\cos m'x\cos n'y\\ &\quad\times\sum_{m''\geq 1,n''\geq 1}\frac{h(t,m'',n'')}{m''}\sin m''x\sin n''y\\ &\quad+\sum_{m'\geq 1} m' f(t,m')\cos m'x\sum_{m''\geq 1,n''\geq 1}\frac{h(t,m'',n'')}{m''}\sin m''x\sin n''y\\ &= \frac{1}{4}\sum_{\substack{m'\geq 1,n'\geq 1\\ m''\geq 1,n''\geq 1}} h(t,m',n')h(t,m'',n'')\cdot\frac{(m')^2+(n')^2}{m''n'}\\ &\quad\times\Big(\sin(m'+m'')x-\sin(m'-m'')x\Big)\cdot\Big(\sin(n'+n'')y-\sin(n'-n'')y\Big)\end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{m'\geq 1\\ m''\geq 1,n''\geq 1}} f(t,m')h(t,m'',n'')\frac{m'}{m''}$$
$$\times\Big(\sin(m'+m'')x-\sin(m'-m'')x\Big)\sin n''y, \tag{1.10}$$

where we have used the trigonometric formula

$$\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\sin(\alpha-\beta).$$

Similarly we compute

$$u_2\cdot\frac{\partial\omega}{\partial y}=-\frac{1}{4}\sum_{\substack{m'\geq 1,n'\geq 1\\ m''\geq 1,n''\geq 1}} h(t,m',n')h(t,m'',n'')\cdot\frac{(m')^2+(n')^2}{m'n''}$$
$$\times\Big(\sin(m'+m'')x+\sin(m'-m'')x\Big)\cdot\Big(\sin(n'+n'')y+\sin(n'-n'')y\Big)$$
$$-\frac{1}{2}\sum_{\substack{m'\geq 1,n'\geq 1\\ m''\geq 1}} h(t,m',n')f(t,m'')\frac{(m')^2+(n')^2}{m'm''}$$
$$\times\Big(\sin(m'+m'')x+\sin(m'-m'')x\Big)\sin n'y. \tag{1.11}$$

Adding together (1.10), (1.11) and re-indexing, we obtain

$$u_1\cdot\frac{\partial\omega}{\partial x}+u_2\cdot\frac{\partial\omega}{\partial y}=\sum_{m\geq 1,n\geq 1}\sin mx\sin ny\cdot\left(\frac{1}{4}\sum_{\substack{m',n',m'',n''\geq 1\\ m'\pm m''=\pm m\\ n'\pm n''=\pm n}} h(t,m',n')h(t,m'',n'')\right.$$
$$\times\left(\frac{(m')^2+(n')^2}{m''n'}\cdot(-1)^{\nu_1(m',m'',m)+\nu_1(n',n'',n)}\right.$$
$$\left.-\frac{(m')^2+(n')^2}{m'n''}\cdot(-1)^{\nu_2(m',m'',m)+\nu_2(n',n'',n)}\right)$$
$$+\frac{1}{2}\sum_{\substack{m',m''\geq 1\\ m'\pm m''=\pm m}} f(t,m')h(t,m'',n)\frac{m'}{m''}\cdot(-1)^{\nu_1(m',m'',m)}$$
$$\left.-\frac{1}{2}\sum_{\substack{m',m''\geq 1\\ m'\pm m''=\pm m}} f(t,m'')h(t,m',n)\frac{(m')^2+n^2}{m'm''}\cdot(-1)^{\nu_2(m',m'',m)}\right). \tag{1.12}$$

Here ν_1, ν_2 are integer functions defined as

$$\nu_1(m',m'',m)=\begin{cases}1, & \text{if } m'-m''=m,\\ 0, & \text{otherwise.}\end{cases}$$

$$\nu_2(m', m'', m) = \begin{cases} 1, & \text{if } m' - m'' = -m, \\ 0, & \text{otherwise.} \end{cases}$$

The formula (1.12) is not enough for giving the equations for $h(t, m, n)$, $f(t, m)$ and an additional step is needed. We remark that the sequence $\{\cos nx,\ n \geq 0\}$ is an orthogonal basis in the Hilbert space $L^2([0, \pi], dx)$. Therefore we can write

$$\sin nx = \sum_{\tilde{n} \geq 0} \Gamma(n, \tilde{n}) \cos \tilde{n}x, \tag{1.13}$$

where

$$\Gamma(n, \tilde{n}) = \begin{cases} \frac{4n}{(n^2 - \tilde{n}^2)\pi}, & \text{if } n - \tilde{n} \text{ is odd}, n \geq 1, \tilde{n} \geq 1 \\ \frac{2}{n\pi}, & \text{if } n \geq 1, \tilde{n} = 0 \text{ and } n \text{ is odd} \\ 0, & \text{otherwise.} \end{cases} \tag{1.14}$$

Return back to (1.12). In the product $\sin mx \cdot \sin ny$ we replace the second factor by the series (1.13). Then the LHS can be written as a series w.r.t. $\sin mx \cdot \cos ny$. The equality between the coefficients of the series on the LHS and RHS gives the needed system of equations for $h(t, m, n)$, $f(t, m)$. Collecting the formulas (1.1), (1.9), (1.12) and (1.13), we obtain

$$\begin{cases} \dot{h}(t, m, n) + \frac{mn}{m^2 + n^2} N(t, m, n) = -(m^2 + n^2) h(t, m, n), & \forall m, n \geq 1, \\ \dot{f}(t, m) + N(t, m, 0) = -m^2 f(t, m), & \forall m \geq 1. \end{cases} \tag{1.15}$$

Here for $m \geq 1$, $n \geq 0$,

$$\begin{aligned} N(t, m, n) = \sum_{k \geq 1} \Gamma(k, n) \cdot \Bigg(& \frac{1}{4} \sum_{\substack{m', n', m'', n'' \geq 1 \\ m' \pm m'' = \pm m \\ n' \pm n'' = \pm k}} h(t, m', n') h(t, m'', n'') \\ & \times \left(\frac{(m')^2 + (n')^2}{m'' n'} \cdot (-1)^{\nu_1(m', m'', m) + \nu_1(n', n'', k)} \right. \\ & \left. - \frac{(m')^2 + (n')^2}{m' n''} \cdot (-1)^{\nu_2(m', m'', m) + \nu_2(n', n'', k)} \right) \\ & + \frac{1}{2} \sum_{\substack{m', m'' \geq 1 \\ m' \pm m'' = \pm m}} f(t, m') h(t, m'', k) \frac{m'}{m''} \cdot (-1)^{\nu_1(m', m'', m)} \\ & - \frac{1}{2} \sum_{\substack{m', m'' \geq 1 \\ m' \pm m'' = \pm m}} f(t, m'') h(t, m', k) \frac{(m')^2 + k^2}{m' m''} \cdot (-1)^{\nu_2(m', m'', m)} \Bigg). \end{aligned} \tag{1.16}$$

The RHS of (1.15) describes the influence of viscosity. The system of (1.15) is our basic ODE system for the coefficients $h(t, m, n)$, $f(t, m)$. The main result of this paper is the following

Theorem 1.1 (Wellposedness and mixed decay) *Let $h(0,m,n)$, $f(0,m)$ satisfy the inequalities*

$$\begin{aligned} |h(0,m,n)| &\le \frac{D_0}{m^\alpha n^\beta}\cdot\frac{mn}{m^2+n^2}, \quad \forall m,n\ge 1,\\ |f(0,m)| &\le \frac{D_0}{m^\alpha}, \quad \forall m\ge 1, \end{aligned} \tag{1.17}$$

and $\alpha>2$, $2<\beta<3$, $D_0>0$. Then there exists a time $T=T(D_0,\alpha,\beta)>0$, a constant $D_1=D_1(D_0,\alpha,\beta)>0$, such that (1.16) *has a unique solution $h(t,m,n)$, $f(t,m)$ which satisfies for all $0\le t<T$ the inequalities*

$$\begin{aligned} |h(t,m,n)| &\le \frac{D_1}{m^\alpha n^\beta}\cdot e^{-\frac{m}{2}t}\cdot\frac{mn}{m^2+n^2}, \quad \forall m,n\ge 1\\ |f(t,m)| &\le \frac{D_1}{m^\alpha}e^{-\frac{m}{2}t}, \quad \forall m\ge 1. \end{aligned} \tag{1.18}$$

In fact $h(t,m,n)$ satisfies an even stronger inequality. For any $0<t_0<T$, there exists a constant $D_2=D_2(D_0,\alpha,\beta,t_0)>0$ such that for any $t_0\le t<T$, we have

$$|h(t,m,n)|\le\frac{D_2}{n^5}e^{-\frac{m}{10}t}, \quad \forall m,n\ge 1. \tag{1.19}$$

Finally if D_0 is sufficiently small, then the corresponding solution is global and the estimates (1.18), (1.19) *hold for $T=+\infty$.*

Remark 1.2 By Theorem 1.1 and especially the decay estimate (1.18), (1.19), we obtain a classical solution to (1.1) satisfying aforementioned boundary conditions. By Sobolev embedding and trace theorem for Lipschitz domains (or directly using the trigonometric expansion), it is not difficult to check that the vorticity ω has continuous second order derivatives up to the boundary. Furthermore, we have $\omega|_{x=0}=\omega|_{x=\pi}=0$ and $\partial_y\omega|_{y=0}=\partial_y\omega|_{y=\pi}=0$. In other words, our boundary conditions (1.4)–(1.5) is satisfied naturally.

Remark 1.3 The uniqueness of solutions to the original 2D NSS system (1.1) with boundary conditions (1.3)–(1.5) essentially follows from the uniqueness of solutions to the ODE system (1.15). Namely fix α, β to be the same as in Theorem 1.1, we introduce the Banach space consisting of functions $\tilde\omega=\tilde\omega(t,x,y)$, such that

$$\tilde\omega(t,x,y)=\sum_{m\ge1,\,n\ge1}\tilde h(t,m,n)\cdot\frac{m^2+n^2}{mn}\cdot\sin mx\cos ny+\sum_{m\ge1}\tilde f(t,m)\sin mx$$

and endowed with the norm

$$\|\tilde\omega\|_{\alpha,\beta}:=\sup_{0\le t\le T}\max\left\{\sup_{m\ge1,n\ge1}|\tilde h(t,m,n)|\cdot m^\alpha n^\beta,\ \sup_{m\ge1}|\tilde f(t,m)|\cdot m^\alpha\right\}.$$

By direct computation or using the proof of Theorem 1.1, it is then not difficult to check that for the same initial data there exists at most one solution to (1.1) in this space.

2 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Let $h(0, m, n)$ satisfy (1.17). The system (1.15) in the integral form can be written as

$$\begin{cases} h(t,m,n) = e^{-(m^2+n^2)t} h(0,m,n) - \int_0^t \frac{mn}{m^2+n^2} e^{-(m^2+n^2)s} N(t-s,m,n)ds, & \forall m,n \geq 1 \\ f(t,m) = e^{-m^2 t} f(0,m) - \int_0^t e^{-m^2 s} N(t-s,m,0)ds\,, & \forall m \geq 1. \end{cases} \tag{2.1}$$

Note here the terms $N(t,m,n)$, $n \geq 0$ are nonlinear functionals of $h(t,m,n)$, $f(t,m)$, $m \geq 1$, $n \geq 1$. We then seek a solution of (2.1) by iterations. Define the iterates

$$\begin{aligned} h^{(1)}(t,m,n) &= e^{-(m^2+n^2)t} h(0,m,n), \quad \forall m,n \geq 1, \\ f^{(1)}(t,m) &= e^{-m^2 t} f(0,m), \quad \forall m \geq 1, \end{aligned} \tag{2.2}$$

and for $j \geq 2$

$$\begin{aligned} h^{(j)}(t,m,n) &= e^{-(m^2+n^2)t} h(0,m,n) \\ &\quad - \int_0^t \frac{mn}{m^2+n^2} e^{-(m^2+n^2)s} N^{(j-1)}(t-s,m,n)ds, \quad \forall m,n \geq 1 \\ f^{(j)}(t,m) &= e^{-m^2 t} f(0,m) - \int_0^t e^{-m^2 s} N^{(j-1)}(t-s,m,0)ds, \quad \forall m \geq 1. \end{aligned} \tag{2.3}$$

Here for $m \geq 1$, $n \geq 0$,

$$\begin{aligned} N^{(j-1)}(t,m,n) = \sum_{k \geq 1} \Gamma(k,n) \cdot \Bigg(& \frac{1}{4} \sum_{\substack{m',n',m'',n'' \geq 1 \\ m' \pm m'' = \pm m \\ n' \pm n'' = \pm k}} h^{(j-1)}(t,m',n') h^{(j-1)}(t,m'',n'') \\ & \times \left(\frac{(m')^2 + (n')^2}{m'' n'} \cdot (-1)^{\nu_1(m',m'',m) + \nu_1(n',n'',k)} \right. \\ & \left. - \frac{(m')^2 + (n')^2}{m' n''} \cdot (-1)^{\nu_2(m',m'',m) + \nu_2(n',n'',k)} \right) \\ & + \frac{1}{2} \sum_{\substack{m',m'' \geq 1 \\ m' \pm m'' = \pm m}} f^{(j-1)}(t,m') h^{(j-1)}(t,m'',k) \frac{m'}{m''} \cdot (-1)^{\nu_1(m',m'',m)} \\ & - \frac{1}{2} \sum_{\substack{m',m'' \geq 1 \\ m' \pm m'' = \pm m}} f^{(j-1)}(t,m'') h^{(j-1)}(t,m',k) \frac{(m')^2 + k^2}{m' m''} \cdot (-1)^{\nu_2(m',m'',m)} \Bigg). \end{aligned} \tag{2.4}$$

We begin with the following lemma.

Lemma 2.1 *Let* $0 < \gamma < \infty$, $n \geq 0$. *There exists a constant* $C_1 > 0$ *depending only on* γ, *such that*

$$\sum_{\substack{k \neq n \\ k \geq 1}} |\Gamma(k,n)| \cdot \frac{1}{k^\gamma} \leq \begin{cases} C_1, & \text{if } 0 \leq n \leq 4 \\ C_1 \cdot \frac{\log n}{n^\gamma}, & \text{if } 0 < \gamma \leq 2, n \geq 5 \\ C_1 \cdot \frac{1}{n^2}, & \text{if } \gamma > 2, n \geq 5. \end{cases} \tag{2.5}$$

Here $\Gamma(k,n)$ *is defined in* (1.14).

Proof In this proof we shall denote by C_1 the constant which may vary from line to line but depends only on γ. By (1.14), we have

$$\sum_{\substack{k \neq n \\ k \geq 1}} |\Gamma(k,n)| \cdot \frac{1}{k^\gamma} \leq C_1 \sum_{\substack{k \neq n \\ k \geq 1}} \frac{k}{|n^2 - k^2|} \cdot \frac{1}{k^\gamma}. \tag{2.6}$$

Consider first the case $0 \leq n \leq 4$. If $1 \leq k \leq 8$ and $k \neq n$, then $|n^2 - k^2| \geq 1 \geq \frac{k^2}{100}$. If $k \geq 9$, then $|n^2 - k^2| \geq \frac{3}{4}k^2 \geq \frac{k^2}{100}$. Therefore if $0 \leq n \leq 4$, then

$$|\text{RHS of (2.6)}| \leq C_1 \sum_{k \geq 1} \frac{1}{k^{1+\gamma}} \leq C_1. \tag{2.7}$$

Next consider the case $n \geq 5$. If $k \leq \frac{n}{2}$, then

$$\begin{aligned} \sum_{1 \leq k \leq \frac{n}{2}} \frac{k}{|n^2 - k^2|} \cdot \frac{1}{k^\gamma} &\leq C_1 \cdot n^{-2} \cdot \sum_{1 \leq k \leq \frac{n}{2}} \frac{1}{k^{\gamma - 1}} \\ &\leq \begin{cases} C_1 \cdot n^{-2}, & \text{if } \gamma > 2 \\ C_1 \cdot n^{-2} \log n, & \text{if } \gamma = 2 \\ C_1 \cdot n^{-\gamma}, & \text{if } 0 < \gamma < 2. \end{cases} \end{aligned} \tag{2.8}$$

If $k \geq 2n$, then

$$\sum_{k \geq 2n} \frac{k}{|n^2 - k^2|} \cdot \frac{1}{k^\gamma} \leq C_1 \sum_{k \geq 2n} \frac{1}{k^{1+\gamma}} \leq C_1 \cdot n^{-\gamma}. \tag{2.9}$$

If $\frac{n}{2} \leq k \leq 2n$, then

$$\begin{aligned} \sum_{\substack{\frac{n}{2} \leq k \leq 2n \\ k \neq n}} \frac{k}{|n^2 - k^2|} \cdot \frac{1}{k^\gamma} &\leq C_1 \cdot n^{-\gamma} \cdot \sum_{\substack{\frac{n}{2} \leq k \leq 2n \\ k \neq n}} \frac{1}{|k - n|} \\ &\leq C_1 \cdot n^{-\gamma} \log n. \end{aligned} \tag{2.10}$$

Collecting the estimates (2.7)–(2.10), we obtain (2.5). The lemma is proved. □

Lemma 2.2 *Let* $\alpha_1 > 1, \alpha_2 > 1$. *Let* $k \geq 1$ *be an integer. There is a constant* $C_2 > 0$ *depending only on* (α_1, α_2) *such that*

$$\sum_{\substack{k_1 \geq 1,\, k_2 \geq 1 \\ |k_1 \pm k_2| = k}} \frac{1}{k_1^{\alpha_1}} \cdot \frac{1}{k_2^{\alpha_2}} \leq C_2 \cdot \frac{1}{k^{\alpha_3}}, \tag{2.11}$$

where $\alpha_3 = \min\{\alpha_1, \alpha_2\}$.

Proof Again in this proof we shall denote by C_2 the constant which may vary from line to line but depends only on γ. Without loss of generality assume $\alpha_1 \leq \alpha_2$. We split the sum on the LHS of (2.11) into two parts. In the first part $k_1 \leq \frac{k}{3}$. Then by $|k_1 \pm k_2| = k$ we get $k_2 \geq \frac{k}{2}$. Therefore

$$\sum_{\substack{1 \leq k_1 \leq \frac{k}{3}, k_2 \geq 1 \\ |k_1 \pm k_2| = k}} \frac{1}{k_1^{\alpha_1}} \cdot \frac{1}{k_2^{\alpha_2}} \leq \frac{C_2}{k^{\alpha_2}} \sum_{k_1 \geq 1} \frac{1}{k_1^{\alpha_1}} \leq \frac{C_2}{k^{\alpha_2}} \leq \frac{C_2}{k^{\alpha_1}}.$$

In the second part $k_1 \geq \frac{k}{3}$. Then

$$\sum_{\substack{k_1 \geq \frac{k}{3}, k_2 \geq 1 \\ |k_1 \pm k_2| = k}} \frac{1}{k_1^{\alpha_1}} \cdot \frac{1}{k_2^{\alpha_2}} \leq \frac{C_2}{k^{\alpha_2}} \sum_{k_2 \geq 1} \frac{1}{k_2^{\alpha_2}} \leq \frac{C_2}{k^{\alpha_1}}.$$

Putting the above two estimates together we obtain (2.11). □

Lemma 2.3 *Let* $\alpha > 2$, $\beta > 2$. *Let* $m \geq 1$, $k \geq 1$ *be integers. There exists a constant* $C_3 > 0$ *depending only on* (α, β) *such that*

$$\sum_{\substack{m', n', m'', n'' \geq 1 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha}(n')^{\beta}} \cdot \frac{1}{(m'')^{\alpha}(n'')^{\beta}} \cdot \frac{m'n''}{(m'')^2 + (n'')^2} \leq C_2 \cdot \frac{1}{m^{\alpha-1}k^{\beta}}, \tag{2.12}$$

$$\sum_{\substack{m', n', m'', n'' \geq 1 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha}(n')^{\beta}} \cdot \frac{1}{(m'')^{\alpha}(n'')^{\beta}} \cdot \frac{m''n'}{(m'')^2 + (n'')^2} \leq C_2 \cdot \frac{1}{m^{\alpha}k^{\beta-1}}. \tag{2.13}$$

Proof We first deal with (2.12). By Lemma 2.2,

$$\begin{aligned}
|\text{LHS of } (2.12)| &\leq \sum_{\substack{m', n', m'', n'' \geq 1 \\ |m' \pm m''| = m \\ |n' \pm n''| = k}} \frac{1}{(m')^{\alpha-1}(n')^{\beta}} \cdot \frac{1}{(m'')^{\alpha}(n'')^{\beta-1}} \cdot \frac{1}{(m'')^2 + (n'')^2} \\
&\leq \sum_{\substack{m', m'' \geq 1 \\ |m' \pm m''| = m}} \frac{1}{(m')^{\alpha-1}(m'')^{\alpha}} \cdot \sum_{\substack{n', n'' \geq 1 \\ |n' \pm n''| = k}} \frac{1}{(n')^{\beta}(n'')^{\beta+1}} \leq C_2 \cdot \frac{1}{m^{\alpha-1}k^{\beta}}.
\end{aligned}$$

Similarly

$$|\text{LHS of (2.13)}| \le \sum_{\substack{m',n',m'',n''\ge 1\\ |m'\pm m''|=m\\ |n'\pm n''|=k}} \frac{1}{(m')^{\alpha}(n')^{\beta-1}}\cdot\frac{1}{(m'')^{\alpha-1}(n'')^{\beta}}\cdot\frac{1}{(m'')^2+(n'')^2}$$

$$\le \sum_{\substack{m',m''\ge 1\\ |m'\pm m''|=m}} \frac{1}{(m')^{\alpha}(m'')^{\alpha+1}}\cdot \sum_{\substack{n',n''\ge 1\\ |n'\pm n''|=k}} \frac{1}{(n')^{\beta-1}(n'')^{\beta}} \le C_2\cdot\frac{1}{m^{\alpha}k^{\beta-1}}. \qquad \square$$

For any $\alpha > 2$, $\beta > 2$, $T > 0$, introduce the Banach space $X_{\alpha,\beta,T}$ consisting of functions $(\tilde h(t), \tilde f(t))$, $\tilde h(t) = (\tilde h(t,m,n))_{m,n\ge 1}$, $\tilde f(t) = (\tilde f(t,m))_{m\ge 1}$ endowed with the norm

$$\left\|(\tilde h(t), \tilde f(t))\right\|_{X_{\alpha,\beta,T}} := \max\left\{ \sup_{0\le t\le T}\sup_{m,n\ge 1} |\tilde h(t,m,n)|\cdot\frac{m^2+n^2}{mn}\cdot m^{\alpha}n^{\beta}\cdot e^{\frac{1}{2}mt},\right.$$
$$\left.\sup_{0\le t\le T}\sup_{m\ge 1}|\tilde f(t,m)|\cdot m^{\alpha}\cdot e^{\frac{1}{2}mt}\right\}.$$

We will prove Theorem 1.1 by a contraction argument in the space $X_{\alpha,\beta,T}$ for some sufficiently small $T > 0$.

Proof of Theorem 1.1 Define the iterations according to (2.2), (2.3). We first show that if T is sufficiently small depending on (D_0, α, β), then

$$\left\|(h^{(j)}(t,m,n),\ f^{(j)}(t,m))\right\|_{X_{\alpha,\beta,T}} \le 2D_0, \quad \forall j \ge 1. \tag{2.14}$$

By (1.17), we have

$$\left\|(h^{(1)}(t,m,n),\ f^{(1)}(t,m))\right\|_{X_{\alpha,\beta,T}} \le D_0.$$

Assume (2.14) holds for $1 \le j < j_0$, $j_0 \ge 2$. Then for $j = j_0$, $n \ge 0$, we have

$$|N^{(j_0-1)}(t,m,n)| \le \sum_{k\ge 1}|\Gamma(k,n)|\cdot\left(\sum_{\substack{m',n',m'',n''\ge 1\\ |m'\pm m''|=m\\ |n'\pm n''|=k}} |h^{(j_0-1)}(t,m',n')|\cdot|h^{(j_0-1)}(t,m'',n'')|\right.$$
$$\times\,((m')^2+(n')^2)\cdot\left(\frac{1}{m''n'}+\frac{1}{m'n''}\right)$$
$$+\sum_{\substack{m',m''\ge 1\\ |m'\pm m''|=m}} |f^{(j_0-1)}(t,m')|\cdot|h^{(j_0-1)}(t,m'',k)|\cdot\frac{m'}{m''}$$
$$\left.+\sum_{\substack{m',m''\ge 1\\ |m'\pm m''|=m}} |f^{(j_0-1)}(t,m'')|\cdot|h^{(j_0-1)}(t,m',k)|\cdot\frac{(m')^2+k^2}{m'm''}\right)$$

$$
\begin{aligned}
&\le 4D_0^2 \sum_{k\ge 1} |\Gamma(k,n)| \\
&\quad \times \Bigg(\sum_{\substack{m',n',m'',n''\ge 1\\ |m'\pm m''|=m\\ |n'\pm n''|=k}} \frac{e^{-\frac{t}{2}(m'+m'')}}{(m')^\alpha (n')^\beta} \cdot \frac{1}{(m'')^\alpha (n'')^\beta} \cdot \frac{m'n''+m''n'}{(m'')^2+(n'')^2} \\
&\quad + \sum_{m',m''\ge 1 |m'\pm m''|=m} \frac{e^{-\frac{t}{2}(m'+m'')}}{(m')^\alpha (m'')^\alpha k^\beta} \cdot \frac{m'k}{(m'')^2+k^2} \\
&\quad + \sum_{m',m''\ge 1 |m'\pm m''|=m} \frac{e^{-\frac{t}{2}(m'+m'')}}{(m')^\alpha (m'')^\alpha k^\beta} \cdot \frac{k}{m''} \Bigg). \qquad (2.15)
\end{aligned}
$$

Now remark that if $|m'\pm m''|=m$, then $m'+m''\ge m$. Therefore by Lemmas 2.3, 2.2, 2.1, we have

$$
\begin{aligned}
|\text{RHS of (2.15)}| &\le 4D_0^2 \cdot e^{-\frac{t}{2}m} \cdot \sum_{k\ge 1} |\Gamma(k,n)| \cdot C_3 \cdot \left(\frac{1}{m^{\alpha-1}k^\beta} + \frac{1}{m^\alpha k^{\beta-1}} \right) \\
&\le C_4 \cdot e^{-\frac{t}{2}m} \cdot \left(\frac{1}{m^{\alpha-1}(n+1)^\beta} + \frac{\log(n+5)}{m^\alpha (n+1)^{\beta-1}} \right),
\end{aligned}
$$

where C_3, C_4 are constants depending only on (D_0,α,β). Substituting this estimate into (2.3), we get

$$
\begin{aligned}
|h^{(j_0)}(t,m,n)| &\le e^{-(m^2+n^2)t} \cdot \frac{mn}{m^2+n^2} \cdot \frac{D_0}{m^\alpha n^\beta} \\
&\quad + \frac{mn}{m^2+n^2} \cdot C_4 \cdot \int_0^t e^{-(m^2+n^2)s} \cdot e^{-\frac{1}{2}(t-s)m} ds \\
&\quad \times \left(\frac{1}{m^{\alpha-1}(n+1)^\beta} + \frac{\log(n+5)}{m^\alpha (n+1)^{\beta-1}} \right) \\
&\le e^{-(m^2+n^2)t} \cdot \frac{mn}{m^2+n^2} \cdot \frac{D_0}{m^\alpha n^\beta} \\
&\quad + \frac{mn}{m^2+n^2} \cdot C_4 \cdot e^{-\frac{t}{2}m} \int_0^t e^{-(m^2/2+n^2)s} ds \\
&\quad \times \left(\frac{1}{m^{\alpha-1}(n+1)^\beta} + \frac{\log(n+5)}{m^\alpha (n+1)^{\beta-1}} \right) \\
&\le e^{-(m^2+n^2)t} \cdot \frac{mn}{m^2+n^2} \cdot \frac{D_0}{m^\alpha n^\beta} + e^{-\frac{t}{2}m} \cdot \frac{mn}{m^2+n^2} \cdot C_4 \\
&\quad \times \frac{1}{(m^2/2+n^2)^{\frac{2}{3}}} \cdot \frac{1-e^{-(m^2/2+n^2)t}}{(m^2/2+n^2)^{\frac{1}{3}} t^{\frac{1}{3}}} \\
&\quad \times t^{\frac{1}{3}} \cdot \left(\frac{1}{m^{\alpha-1}(n+1)^\beta} + \frac{\log(n+5)}{m^\alpha (n+1)^{\beta-1}} \right). \qquad (2.16)
\end{aligned}
$$

Now remark that

$$\sup_{x>0} \frac{1-e^{-x}}{x^{\frac{1}{3}}} \le C_5 < \infty,$$

where C_5 is an absolute constant. Also it is clear that

$$\frac{1}{(m^2/2+n^2)^{\frac{2}{3}}} \cdot \left(\frac{1}{m^{\alpha-1}(n+1)^{\beta}} + \frac{\log(n+5)}{m^{\alpha}(n+1)^{\beta-1}} \right) \le C_6 \cdot \frac{1}{m^{\alpha}} \cdot \frac{1}{n^{\beta}},$$

where C_6 is another constant depending only on (D_0, α, β). Substituting the above two estimates into the RHS of (2.16), we obtain

$$|h^{(j_0)}(t,m,n)| \le e^{-\frac{t}{2}m} \cdot \frac{mn}{m^2+n^2} \cdot \frac{1}{m^{\alpha}n^{\beta}} \cdot (D_0 + t^{\frac{1}{3}} \cdot C_4 \cdot C_5 \cdot C_6).$$

Let $T \le \frac{D_0^3}{C_4^3 \cdot C_5^3 \cdot C_6^3}$. Then for $0 < t < T$ we get

$$|h^{(j_0)}(t,m,n)| \le e^{-\frac{t}{2}m} \cdot \frac{mn}{m^2+n^2} \cdot \frac{1}{m^{\alpha}n^{\beta}} \cdot 2D_0. \tag{2.17}$$

By a similar computation, we also obtain

$$\begin{aligned} |f^{(j_0)}(t,m)| &\le e^{-m^2 t}|f(0,m)| + \int_0^t e^{-m^2 s} \cdot e^{-\frac{m}{2}(t-s)} \cdot \frac{C_7}{m^{\alpha-1}} ds \\ &\le e^{-\frac{m}{2}t} \cdot \frac{D_0}{m^{\alpha}} + e^{-\frac{m}{2}t} \cdot \frac{C_8}{m^{\alpha}} \cdot t^{\frac{1}{3}}, \end{aligned}$$

where C_7 and C_8 are constants depending only on (D_0, α, β). Taking $T \le \min\{\frac{D_0^3}{C_4^3 \cdot C_5^3 \cdot C_6^3}, \frac{D_0^3}{C_8^3}\}$, we get for $0 < t < T$,

$$|f^{(j_0)}(t,m)| \le e^{-\frac{m}{2}t} \cdot \frac{2D_0}{m^{\alpha}}. \tag{2.18}$$

Clearly (2.17), (2.18) implies that (2.14) holds for $j = j_0$. This finishes the induction step and (2.14) is proved. By essentially repeating the above estimates, we also obtain strong contraction of the sequence $(h^{(j)}(t), f^{(j)}(t))$. Namely there exists $T_0 = T_0(D_0, \alpha, \beta) > 0$ and a constant $0 < \theta < 1$, such that if $T \le T_0$ then

$$\begin{aligned} &\left\| (h^{(j+1)}(t), f^{(j+1)}(t)) - (h^{(j)}(t), f^{(j)}(t)) \right\|_{X_{\alpha,\beta,T}} \\ &\quad \le \theta \cdot \left\| (h^{(j)}(t), f^{(j)}(t)) - (h^{(j-1)}(t), f^{(j-1)}(t)) \right\|_{X_{\alpha,\beta,T}}, \quad \forall j \ge 2. \end{aligned}$$

This shows that $(h^{(j)}(t), f^{(j)}(t))$ is Cauchy in $X_{\alpha,\beta,T}$ and hence we have shown the existence and uniqueness of a solution to (2.1) in $X_{\alpha,\beta,T}$. Consequently (1.18) holds with $D_1 = 2D_0$. We still have to show (1.19). We shall establish this by a bootstrap argument. Without loss of generality assume $t_0 < \frac{T}{100}$. Denote $\hat{h}(t) = h(t + \frac{t_0}{2})$, $\hat{f}(t) = f(t + \frac{t_0}{2})$. Then $(\hat{h}(t), \hat{f}(t))$ solves (2.1) with $(h(\frac{t_0}{2}), f(\frac{t_0}{2}))$ as initial data. For $0 \le t < T - \frac{t_0}{2}$, by (1.18), we get

$$|\hat{h}(t,m,n)| = \left|h\left(t + \frac{t_0}{2}, m, n\right)\right| \le \frac{D_1}{m^\alpha n^\beta} \cdot e^{-\frac{m}{2}t} \cdot e^{-\frac{m}{4}t_0} \cdot \frac{mn}{m^2+n^2}$$
$$\le D_1 \cdot e^{-\frac{m}{2}t} \cdot \frac{1}{n^{\beta+1}} \cdot \frac{e^{-\frac{m}{4}t_0}}{m^{\alpha-1}}.$$

Since $m \ge 1$ and $t_0 > 0$, we have

$$D_1 \cdot \sup_{m\ge 1} \frac{e^{-\frac{m}{4}t_0}}{m^{\alpha-1}} \le D_3 < \infty,$$

where D_3 is a constant depending only on $(D_0, \alpha, \beta, t_0)$. Therefore we obtain

$$|\hat{h}(t,m,n)| \le D_3 \cdot e^{-\frac{m}{2}t} \cdot e^{-\frac{m}{6}t_0} \cdot \frac{1}{n^{\beta+1} m^{10}}, \quad \forall m, n \ge 1. \tag{2.19}$$

Similarly we have

$$|\hat{f}(t,m)| \le D_3 \cdot e^{-\frac{m}{2}t} \cdot e^{-\frac{m}{6}t_0} \cdot \frac{1}{m^{10}}, \quad \forall m \ge 1, \tag{2.20}$$

where D_3' is a constant depending only on $(D_0, \alpha, \beta, t_0)$. Denote $\hat{N}(t)$ as the expression in (1.16) with $(h(t), f(t))$ now replaced by $(\hat{h}(t), \hat{f}(t))$. Then by Lemmas 2.2, 2.3, 2.1 and (2.19), (2.20), we have for any $0 \le s < T - \frac{t_0}{2}$,

$$|\hat{N}(s,m,n)| \le e^{-\frac{m}{2}s - \frac{m}{6}t_0} \cdot D_4 \cdot \frac{\log(n+5)}{(n+1)^{\beta-1}},$$

where D_4 is another constant depending only on $(D_0, \alpha, \beta, t_0)$. Plugging this estimate into the RHS of (2.1) and using again (2.19) for $\hat{h}(0,m,n)$, we obtain for $t \ge \frac{t_0}{100}$,

$$|\hat{h}(t,m,n)| \le e^{-(m^2+n^2)t} |\hat{h}(0,m,n)| + \frac{m}{n} \cdot e^{-\frac{m}{6}t_0} \cdot D_4 \cdot \frac{\log(n+5)}{(n+1)^{\beta-1}} \cdot e^{-\frac{m}{2}t}$$
$$\times \int_0^t e^{-(m^2+n^2-\frac{m}{2})s} ds$$
$$\le e^{-(m^2+n^2)t} \cdot D_3 \cdot e^{-\frac{m}{2}t - \frac{m}{6}t_0} \cdot \frac{1}{n^{\beta+1}m^{10}} + m \cdot e^{-\frac{m}{6}t_0 - \frac{m}{2}t} \cdot D_4 \cdot \frac{\log(n+5)}{n^{\beta+2}}$$
$$\le D_5 \cdot e^{-\frac{m}{2}t - \frac{m}{7}t_0} \cdot \frac{\log(n+5)}{n^{\beta+2}},$$

where D_5 depends only on $(D_0, \alpha, \beta, t_0)$ and we have used the fact that $t \ge \frac{t_0}{100}$ in bounding the linear term. Compare this bound with (2.19), we have a better estimate of $\hat{h}(t,m,n)$ with the decay in n improved from $n^{-(\beta+1)}$ to $n^{-(\beta+2)} \cdot \log(n+5)$. Iterating the above process once more and noting that (2.5) only produces a decay of n^{-2} for $\gamma > 2$, we obtain for any $\frac{t_0}{100} \le s < T - \frac{t_0}{2}$,

$$|\hat{N}(s,m,n)| \le e^{-\frac{m}{2}s - \frac{m}{7}t_0} \cdot D_6 \cdot \frac{1}{n^2},$$

and consequently

$$|\hat{h}(t,m,n)| \le D_7 \cdot e^{-\frac{m}{2}t-\frac{m}{8}t_0} \cdot n^{-5}.$$

Here again D_6, D_7 are constants depending only on $(D_0, \alpha, \beta, t_0)$. Hence (1.19) holds. Finally we remark that the small data result follows along the same lines as in the proof of the local contraction argument. We omit the standard details. The theorem is proved. □

Acknowledgements We thank C. Bardos, G. Seregin, E. Titi, V. Yakhot for numerous useful discussions. Special thanks go to N. Chernov for his collaboration, careful reading of the text and excellent numerical work which he did in connection with this paper. The first author acknowledges the support from the grant NCNIL 07-01-9221. The second author is supported in part by NSF DMS 0635607 and a start-up funding from Mathematics Department of University of Iowa. The third author acknowledges the financial support from NSF grant DMS 0600996.

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